CHAPTER 7

RELIABILITY DATA ANALYSIS

INTRODUCTION

It is important to understand that for any realistic situation the true re-
liability characteristics of a system, or fleet of systems, are never known
with complete certainty. This is true, of course, because we have not, in
fact, actually tested every system in the total population and, practically
speaking, never could. To compensate for this lack of total information, some
form of sampling is used to obtain information about the reliability char-
acteristics inherent in a system and to quantify the level of uncertainty
about them. Of course, uncertainty continues to exist, and, as a consequence,
the reliability parameters can only be estimated. This chapter presents pro-
cedures which can be used to determine estimates for the various reliability
parameters and to quantify the uncertainty inherent in these estimates.

These procedures support the analysis of data gathered in previously conducted
tests. Planning these tests to assure that adequate sample sizes are obtained
is the topic of Chapter 8. The objective of the data analysis effort is to
determine "best estimates" of system performance parameters, such as reli-
ability, and to estimate the uncertainty associated with these "best estimate"
values.

As in previous chapters, the case studies illustrate the application and
manipulation of the mathematical concepts presented in the chapter text. Note
that in the typical Chapter 7 case study, YOU are provided the <u>results</u> of a
hypothetical test program and requested to develop a best estimate and con-
fidence interval for a reliability parameter.

<u>TYPES OF RELIABILITY TESTS</u>

<u>Fixed Configuration and Growth Tests</u>

There are basically two types of reliability tests. One is a test of fixed
configuration. The other is the growth, or developmental, test, which centers
on reliability improvement seen as the configuration changes during the test.
There is not, however, a clean line between these two types. For the truly
fixed configuration test of continuously operated systems, any changes in
reliability are due to the inherent characteristics of the hardware and how it
is ▢aintained. The analysis is done as a function of system age. If there
are design changes, they have to be considered on a separate basis, perhaps by
a data set for each configuration. See Chapter 10 for more details on this
procedure.

For the growth type of test, the statistical models currently available assume
that all changes in reliability are attributable to the design changes. In
other words, they assume that the inherent reliability of the hardware is
constant. The basic analysis for the growth type of test is done as a func-
tion of test exposure, rather than age, since it is test exPosur that pro-
vides information for design improvements. The effects of system age can be

dealt with separately, primarily by considering the failure modes that are observed. Chapter 9 summarizes the topic of reliability growth and illustrates the associated analysis techniques.

Discrete and Continuous Tests

The most elementary consideration in beginning a data analysis is to determine whether test time is measured continuously or discretely. Usually, this distinction is quite obvious. An example of a test which can be analyzed either way is the following. Suppose that a system has a durability requirement of 5000 hours and ten systems are available for testing. Each system is tested until it either experiences a durability failure or successfully completes the 5000 hour test period. We can let each system be a test unit and count as a failure any system which fails before 5000 hours. This is a discrete time approach. Alternatively, we could let hours be our test units, with the total operating hours of the 10 -systems as the test exposure. This is a continuous time approach. Another example is the firing of an automatic weapon, where many rounds are fired. This is a one-shot, discrete time test if we are analyzing the ammunition, but could be considered a continuous time test if we are analyzing the gun or any of its components. Generally, when either approach is appropriate, more information is obtained from the continuous time approach.

DISCRETE TIME TESTING

Suppose that the systems under test are single-shot systems. Each test unit results in a distinguishable success or failure. As discussed in Chapter 5, the binomial model will be used to represent or model system reliability when discrete time or success/fail operations are of interest. It is assumed throughout this discussion on discrete time testing that the conditions of a binomial model are reasonably satisfied. (See Chapter 5.) We present data analysis for success/fail (discrete) tests in the form of point estimates, confidence intervals, and tests of hypotheses.

Binomial Model: Point Estimate of Failure Probability

Once the number of trials has been specified (see Chapter 8), all the information contained in a binomial experiment rests in the number of failures that occur. We use this information to make an assessment or an estimate of the true probability of failure, p. Thus, our best estimate of the value of p is the ratio of the number of failures to the number of trials. This ratio is called the sample proportion of failures and is designated by the symbol $\hat{p}$, called p-hat. We use this sample proportion of failures, $\hat{p}$, to construct confidence intervals for p and in testing hypotheses about p. By definition, then

$$\hat{p} = \frac{\text{number of failures}}{\text{number of trials}} = \text{sample proportion of failures}$$

$$\hat{p} = \text{best estimate for p}$$

$$p = \text{true proportion of failures}$$

Note that true system reliability is the probability of successful operation, therefore

$R = 1 - p$, where R is true system reliability, and

$\hat{R} = 1 - \hat{p}$ = best estimate of system reliability.

It is important that the test designer and/or evaluator understand that a point estimate for p represents a **small** portion of the information contained in the data generated by a binomial experiment. Other useful information includes upper and lower confidence limits for the unknown parameter, p.

<u>Binomial Model: Confidence Limits for Failure Probability</u>

Confidence limits and their interpretation should play a vital role in designing and evaluating a binomial experiment. Not only does the actual interval relay a significant amount of information about the data, but also the method of interval construction can aid the test designer in determining adequate test exposure to meet his needs. **An** extensive discussion on the interpretation of confidence intervals is given in Chapter 6.

Suppose that we observe "s" failures out of "n" trials in a binomial experiment. This translates to a sample proportion of failures equal to s/n and a sample proportion of successes equal to (n-s)/n. Tables of exact confidence limits for the true proportion of failures for values of n less than or equal to 30 are given in Appendix B, Table 4. As an example, suppose that n = 25 trials and s = 4 failures. A 90% upper confidence limit for p is 0.294. We obtain this value using Appendix B, Table 4 with n = 25 in the column labeled 90% upper limit and the row labeled s = 4. For the same data, a 98% confidence interval is

$$0.034 \leq p \leq 0.398.$$

In this case, the values are found in the columns labeled 98% interval and the row labeled s = 4. More examples using Table 4 are given in Case Study 7-3.

<u>Binomial Model: Confidence Levels for Pre-Established Reliability Limits</u>

If, after conducting a test in which we observed s failures (c = n-s **successes**) out of n trials, we wish to determine how confident we are that a pre-established level of reliability (such as the **MAV**) has been met or exceeded, we may use equation 7.1 below.

Let R_L designate the desired pre-established level of reliability. To find the confidence that R_L has been met or exceeded, we evaluate the expression:

$$B_{n,R_L}(c-1) = \sum_{k=0}^{c-1} \binom{n}{k} R_L^{k}(1-R_L)^{n-k} \tag{7.1}$$

If we denote the value of this expression as $1 - \alpha$, then we are $100(1-\alpha)\%$ confident that $R \geq R_L$."

If, on the other hand, we wish to determine how confident we are that a pre-established level of reliability (such as the SV) has not been attained, we may use equation 7.2.

Let R_U designate the desired pre-established level of reliability. To find the confidence that R_U has not been attained, we evaluate the expression:

$$B_{n,R_U}(c) = \sum_{k=0}^{c} \binom{n}{k} R_U^k (1-R_U)^{n-k}$$

(7.2)

If we denote the value of this expression as α, then we are $100(1 - \alpha)\%$ confident that $R \leq R_U$.

See Case Study 7-1 for an example of this technique.

The Greek letter α is used numerous times throughout this chapter to represent a generalized value or designation of "RISK." In this chapter, α is not necessarily to be interpreted as producer's risk as in Chapters 6 and 8.

Approximate Binomial Confidence Limits (Normal Approximation)

If the number of failures and the number of successes both are greater than or equal to 5, we can obtain approximate confidence limits using the normal distribution. The approximate $100(1-\alpha)\%$ lower limit for p, the true proportion of failures, is

$$p \geq p_L$$

$$p \geq \hat{p} - z_\alpha \sqrt{\hat{p}(1-\hat{p})/n} \, ,$$

(7.3)

where $\hat{p} = s/n$. The approximate $100(1-\alpha)\%$ upper confidence limit for p is

$$p \leq p_U$$

$$p \leq \hat{p} + z_\alpha \sqrt{\hat{p}(1-\hat{p})/n} \, .$$

(7.4)

The two-sided $100(1-\alpha)\%$ confidence limits for p are

$$p_L \leq p \leq p_U$$

$$\hat{p} - z_{\alpha/2} \sqrt{\hat{p}(1-\hat{p})/n} \leq p \leq \hat{p} + z_{\alpha/2} \sqrt{\hat{p}(1-\hat{p})/n} \, .$$

(7.5)

Values for z_α and $z_{\alpha/2}$ are obtained from Appendix B, Table 2.

As an example, suppose that 5 failures in 30 trials occurred during a test. An approximate 90% ((Y = 0.10) upper limit for the true proportion of failures is

$$\hat{p} + z_{0.10}\sqrt{\hat{p}(1-\hat{p})/n} \ .$$

Substituting n = 30, and p = 5/30 = 0.166, we obtain

$$0.166 + z_{0.10}\sqrt{(0.166)(0.834)/30} \ .$$

'he value 'f $z_{0.10}$ is determined using Appendix B, Table 2. Under the column labeled $P(Z \geq z_\alpha)$ we search the values until we find the number closest to 0.10, the value of α. The number in the column labeled z_α is the desired value. In this case, for α = 0.10, z_α = 1.28. The upper limit is then

$$0.166 + 1.28\sqrt{(0.166)(0.834)/30} \ ,$$

which reduces to 0.253. We are thus 90% confident that the true proportion of failures is 0.253 or smaller.

See Case Study 7-3 for construction of confidence limits using normal approximation.

<u>Approximate Binomial Confidence Limits (Poisson/Exponential Approximation)</u>

When the sample proportion of failures is small, and the number of **trials** is reasonably large--at least 30--we can obtain approximate confidence limits using techniques described in the section on Exponential Model: Confidence Intervals and Limits for MTBF. This is an especially useful technique for situations involving very few failures in fairly large samples. We use the procedure for <u>failure terminated testing</u> with the identifications: T = n (the number of trials) and r = s (the number of failures). We obtain approximate confidence limits for p, the probability of failure, by constructing confidence limits for θ, the system MTBF. Because p and A are failure-oriented parameters and θ is a success-oriented parameter (remember that by definition θ = 1/A), an approximate confidence limit for p is the reciprocal of the confidence limit for θ. An important consequence of the reciprocity mentioned above is that an upper confidence limit for θ yields a lower confidence limit for p and vice versa.

Consider the situation described in Chapter 6, where 3 failures out of 30 trials of a binomial experiment were observed. To construct an approximate 90% confidence interval for the true proportion of failures, we let T be 30 and r be 3. The 95% confidence interval for θ is

$$\frac{2T}{\chi^2_{\alpha/2,2r}} < \theta \leq \frac{2T}{\chi^2_{1-\alpha/2,2r}}$$

since T = n= 30, r = s = 3, and α = 0.05, we have

$$\frac{2(30)}{\chi^2_{0.025,6}} \leq \theta \leq \frac{2(30)}{\chi^2_{0.975,6}}$$

Values $\chi^2_{0.025,6}$ and $\chi^2_{0.975,6}$ are obtained from Appendix B, Table 5. The explanation of how to extract these values is presented below in the section entitled "Exponential Model: Confidence Intervals and Limits for MTBF." The values are $\chi^2_{0.025,6}$ = 14.46 and $\chi^2_{0.975,6}$ = 1.24. Thus the interval for θ is

$$\frac{2(30)}{14.46} \leq \theta \leq \frac{2(30)}{1.24} ,$$

which, upon simplification, becomes

$$4.15 \leq \theta \leq 48.39 .$$

Taking the reciprocals of the limits for θ, we have that the approximate 95% confidence interval for the true proportion of failures is

$$0.021 \leq p \leq 0.241 .$$

Since reliability is 1 - p, the approximate 95% confidence interval for system reliability is

$$0.759 \leq R \leq 0.979 .$$

This statement can also be interpreted as follows: We are 95% confident that the true system reliability is between 0.759 and 0.979. This interval is based on our test results where 3 out of 30 trials ended in failure.

See Case Study 7-2 for another example of this procedure.

Point Estimates and Confidence Limits for the Difference/Ratio of Proportions

Suppose that tests have been conducted on two different types of systems resulting in sample proportions of failures of $\hat{p}_1$ and $\hat{p}_2$ with sample sizes of n_1 and n_2, respectively. The point estimates for the difference $(p_1 - p_2)$ and ratio (p_1/p_2) of proportions are the difference and ratio of the sample proportions, i.e., $\hat{p}_1 - \hat{p}_2$ and $\hat{p}_1/\hat{p}_2$, respectively. We present the procedures for determining confidence limits for the difference and for the ratio of the two population proportions $(p_1$ and $p_2)$ using the normal distribution. The approximate 100(1 - α)% lower confidence limit for the true difference in proportions is

$$p_1 - p_2 \geq (p_1 - p_2)_L$$

$$\geq \hat{p}_1 - \hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2} .$$

The approximate 100 (1 - α)% upper confidence limit for the true difference in proportions is

$$p_1 - p_2 \le (p_1 - p_2)_U$$

$$\le \hat{p}_1 - \hat{p}_2 + z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2} \ .$$

The approximate 100(1 - α)% confidence interval for the true difference in proportions is

$$(p_1 - p_2)_L \le p_1 - p_2 \le (p_1 - p_2)_U$$

$$\hat{p}_1 - \hat{p}_2 - z_{\alpha/2} \sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2} \le p_1 - p_2$$

$$\le \hat{p}_1 - \hat{p}_2 + z_{\alpha/2} \sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2} \ .$$

With high reliability systems, it is sometimes more informative for comparing two systems to look at the ratio of proportions of failures. As an example, suppose that the true proportions of failures for two systems are 0.01 and 0.001. We can say that one system is ten times better than the other even though the difference is a mere 0.009. An approximate 100(1 - α)% lower confidence limit for the true ratio of proportions is

$$p_1/p_2 \ge (p_1/p_2)_L$$

$$p_1/p_2 \ge \hat{p}_1/\hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

The approximate 100 (1 - α)% upper confidence limit for the true ratio of proportions is

$$p_1/p_2 \le (p_1/p_2)_U$$

$$p_1/p_2 \le \hat{p}_1/\hat{p}_2 + z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

The approximate 100 (1 - α)% confidence interval for the true ratio of proportions is

$$(p_1/p_2)_L \le p_1/p_2 \le (p_1/p_2)_U$$

$$\hat{p}_1/\hat{p}_2 - z_{\alpha/2} \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \le p_1/p_2$$

$$\le \hat{p}_1/\hat{p}_2 + z_{\alpha/2} \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

In Case Study 7-4, we construct confidence limits for the difference and ratio of population proportions.

CONTINUOUS TIME TESTING

Suppose the systems under test operate as a function of hours, kilometers, or other continuous measure.   In such a case, the data are not solely success/ failure oriented.   Generally, the times at which failures occur and the time in operation without failures must also be considered.   These types of tests are analyzed by using a Poisson model.   When the failure rate remains constant throughout the test, the exponential distribution describes the times between failures and provides all the information needed for the data analysis. For the analysis presented in subsequent sections of this chapter, we will assume that the failure rate is constant.   We present below a graphical procedure to determine if that assumption is reasonable.

Continuous Time Testing: Failure Pattern Identification

When confronted with data from a continuous time test the analyzer should first construct an average failure rate plot.   The purpose of constructing an average failure rate plot is to help the analyst determine whether the failure rate is increasing, decreasing, or is constant.   The type of failure rate plot that will be described considers hardware that did not have significant design changes made, so that changes in the failure rate are due primarily to the age of the equipment.   (When substantial design changes are made, there may be reliability growth.   In that case, a different type of average failure rate plot is used, which is based on cumulative test exposure rather than the age of the equipment.)

The average failure rate plot is constructed as follows:

1.   Determine the lowest and highest equipment ages which the test experience covers.   These need not be ages at which failures occurred. This estab- lishes the lower and upper limits of the plot. For convenience, working limits may be set at "round" numbers above and below the lower and upper limits , respectively.

2.   Divide the interval encompassed by the working limits into **subintervals**. The subintervals need not be of equal size.

3.   Count the number of failures in each subinterval.   (A minimum of 5 fail- ures per subinterval is desirable, though not absolutely necessary.)

4.   Add up the hours (or miles, rounds, etc.) of operation within each **sub- interval**.

5.   Compute the average failure rate for each subinterval by dividing the number of failures in the subinterval by the hours (or miles, rounds, etc.) of operation in the **subinterval**.

6.   Construct a graph, with the system age (in hours, miles, rounds, etc.) on the horizontal scale, and failure rate on the vertical scale.   The aver- age failure rates computed for each subinterval are shown as horizontal lines over the length of each subinterval.

7.   If the average failure rate plot has too much fluctuation to show any kind of trend, reduce the number of **subintervals** and repeat steps 3

through 6.   For very small amounts of data, it may be necessary to use only two subintervals.

8.   From the final version of the average failure rate plot, judge whether the failure rate trend remains constant, increases , or decreases as the equipment ages.   For small amounts of data it may be difficult to make this judgment.   **In** any case, statistical tests for trend may be used.

9.   If the data are judged to have no trend, analyses based on the exponential distribution may generally be used with validity.

10.   If the failure rate is judged to be increasing or decreasing, as a minimum, a note to this effect should accompany any analyses based on the assumption of exponential times between failures.   To analyze data that appear to have a trend more explicitly, a non-homogeneous Poisson process may be fitted to the data.   We do not present any analysis using a non-homogeneous Poisson process in this chapter.   If the average failure rate plot indicates that a constant failure rate assumption is unwarranted, the data analyst may refer to a statistics text which covers the topic of stochastic processes in depth to aid in his analysis.

11.   See Case Studies 7-5 and 7-6 for examples of average failure rate plots.

Exponential Model: Point Estimate of MTBF

When data are judged to show a constant failure rate, the exponential distribution may be used for data analysis.   Exponential analysis does not require the use of actual failure times.

Notation T =   total test exposure, the total hours, miles, etc. , accumulated among all the items included in the sample

r =   number of failures observed

$\hat{\theta}$ =   point estimate of MTBF

$\hat{R}(x)$ =   point estimate of reliability for a specified exposure, x

$\hat{\lambda}$ =   the point estimate of the failure rate

Formulas

$$\hat{\theta} = \frac{T}{r}$$
(7.6)

Exponential Model: Point Estimates of Reliability and Failure Rate

Point estimates of reliability and failure rate may be developed from point estimates of MTBF as follows:

$$i(x) = e^{-x/\hat{\theta}}$$
(7.7)

$$\hat{\lambda} = 1/\hat{\theta} \tag{7.8}$$

See Case Studies 7-7, 7-8, and 7-9 **for** illustrations of computing point estimates.

Exponential Model: Confidence Intervals and Limits for MTBF

<u>Notation</u> T =   total test exposure, the total hours, **miles,** etc. , accumulated among all the items included in the sample

r =   the number of failures observed

$X^2_{\underline{\quad},\underline{\quad}}$ =   a **chi-square** value, identified by two subscripts. To determine a **chi-square** value using Appendix B, Table 5, we use the first subscript, a function of the risk (α), to indicate the column, and the second subscript, a function of the number of failures (r), to indicate the row.

α =   the risk that a confidence statement is in error. Note: The symbol a used here does not necessarily represent the producer's risk as discussed in Chapter 6.

θ = MTBF

$R(x)$ = reliability for a period x

λ = failure rate

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right|$ Used in conjunction with $\left\{\begin{matrix} \text{no subscript = true} \\ \text{but unknown value} \\ \\ \text{L subscript = lower limit} \\ \\ \text{U subscript = upper limit} \end{matrix}\right.$

<u>Formulas</u> (All the formulas listed below will yield statements at the 100(1-a%) level of confidence.)

<u>Confidence Limits</u>

<u>Time Terminated</u>	<u>Failure Terminated</u>
When the test exposure ends at a time other than a failure occurrence, use Appendix B, Table 8a multipliers or the following formulas.	When the test exposure ends at a failure occurrence, use Appendix B, Table **8b** multipliers or the following formulas.

Interval for specified confidence level

Interval for specified confidence level

$$\theta_L \le \theta \le \theta_U$$

$$\theta_L \le \theta \le \theta_U$$

$$\frac{2T}{\chi^2_{\alpha/2,2r+2}} \le \theta \le \frac{2T}{\chi^2_{1-\alpha/2,2r}} \quad (7.9a)$$

$$\frac{2T}{\chi^2_{\alpha/2,2r}} \le \theta \le \frac{2T}{\chi^2_{1-\alpha/2,2r}} \quad (7.9b)$$

See Case Studies 7-7 and 7-8.

See Case Study 7-9.

Lower limit for specified confidence level

Lower limit for specified confidence **level**

$$\theta \ge \theta_L$$

$$\theta \ge \theta_L$$

$$\theta \ge \frac{2T}{\chi^2_{\alpha,2r+2}} \quad (7.10a)$$

$$\theta \ge \frac{2T}{\chi^2_{\alpha,2r}} \quad (7.10b)$$

Upper limit for specified confidence level

Upper limit for specified confidence level

$$\theta \le \theta_U$$

$$\theta \le \theta_U$$

$$\theta \le \frac{2T}{\chi^2_{1-\alpha,2r}} \quad (7.11a)$$

$$\theta \le \frac{2T}{\chi^2_{1-\alpha,2r}} \quad (7.11b)$$

See Case Study 7-7,

See Case Study 7-9.

Confidence That A Specific Lower Limit Has Been Attained

Time Terminated	Failure Terminated

Confidence that a specific lower limit, θ_L, has been attained

$$\chi^2_{\alpha,2r+2} = \frac{2T}{\theta_L} \qquad (7.12a)$$

Search χ^2 tables in row labeled $2r + 2$ for the numerical value, $2T/\theta_L$, and find the associated value for α.

Confidence that

$$\theta \geq \theta_L$$

is $100(1-a)\%$.

The value, α, may also be determined in closed form as follows:

$$\alpha = \sum_{k=0}^{r} \frac{(T/\theta_L)^k \, e^{-(T/\theta_L)}}{k!} \qquad (7.13a)$$

(Use Appendix B, Table 3 or Chart 1 to evaluate this expression.)

Confidence that

$$\theta \geq \theta_L$$

is $100(1-a)\%$.

See Case Studies 7-7 and 7-8.

Confidence that a specific lower limit, θ_L, has been attained

$$\chi^2_{\alpha,2r} = \frac{2T}{\theta_L} \qquad (7.12b)$$

Search χ^2 tables in row labeled $2r$ for the numerical value, $2T/\theta_L$, and find the associated value for α.

Confidence that

$$\theta \geq \theta_L$$

is $100(1-CY)\%$.

The value, α, may also be determined in closed form as follows:

$$\alpha = \sum_{k=0}^{r-1} \frac{(T/\theta_L)^k \, e^{-(T/\theta_L)}}{k!} \qquad (7.13b)$$

(Use Appendix B, Table 3 or Chart 1 to evaluate this expression.)

Confidence that

$$\theta \geq \theta_L$$

is $100(1-a)\%$.

See Case Study 7-9.

Confidence That A Specific Upper Limit Has Not Been Attained

Time Terminated

Confidence that a specified upper limit, θ_U, **has** not been attained

$$\chi^2_{1-\alpha,2r} = \frac{2T}{\theta_U} \qquad (7.14a)$$

Search χ^2 tables in the row labeled 2r for the numerical value, $2T/\theta_U$, and find the associated value for 1-α.

Confidence that

$$\theta \leq \theta_U$$

is 100(1-CY)%.

The value α may also be determined in closed form using the following equation:

$$1-\alpha = \sum_{k=0}^{r=1} \frac{(T/\theta_U)^k e^{-(T/\theta_U)}}{k!} \qquad (7.15a)$$

(Use Appendix B, Table 3 or Chart 1 to evaluate this expression.)

Confidence that

$$\theta \leq \theta_U$$

is 100(1-α)%.

See Case Study 7-7.

Failure Terminated

Confidence that a specified upper limit, θ_U, has not been attained

$$\chi^2_{1-\alpha,2r} = \frac{2T}{\theta_U} \qquad (7.14b)$$

Search χ^2 tables in the row labeled 2r for the numerical value, $2T/\theta_U$, and find the associated value for 1-α.

Confidence that

$$\theta \leq \theta_U$$

is 100(1-a)%.

The value α may also be determined in closed form using the following equation:

$$1-\alpha = \sum_{k=0}^{r-1} \frac{(T/\theta_U)^k e^{-(T/OU)}}{k!} \qquad (7.15b)$$

(Use Appendix B, Table 3 or Chart 1 to evaluate this expression.)

Confidence that

$$\theta \leq \theta_U$$

is 100(1-Q)%.

See Case Study 7-9.

Exponential Model: Confidence Intervals and Limits for Reliability and Failure Rate

Intervals for reliability and failure rate with 100(1-a)% confidence are

$$R_L(x) \le R(x) \le R_U(x)$$

$$e^{-(x/\theta_L)} \le R(x) \le e^{-(x/\theta_U)} \qquad (7.16)$$

and

$$\lambda_L \le \lambda \le \lambda_U$$

$$1/\theta_U \le A \le I/e_L \qquad (7.17)$$

where θ_L and 0_u are the lower and upper limits of the 100(1-a)% confidence interval for θ (MTBF).

Lower limit for reliability and upper limit for failure rate with 100(1-α)% confidence are

$$R(x) \ge R_L(x)$$

$$R(x) \ge e^{-(x/eL)} \qquad (7.18)$$

and

$$\lambda \le \lambda_U$$

$$\lambda \le 1/\theta_L \qquad (7.19)$$

where θ_L is the 100(1-α)% lower confidence limit for θ (MTBF).

Upper limit for reliability and lower limit for failure rate with 100(1-α)% confidence are

$$R(x) \le R_U(x)$$

$$R(x) \le e^{-(x/eu)} \qquad (7.20)$$

and

$$\lambda \ge \lambda_L$$

$$\lambda \ge 1/\theta_U \qquad (7.21)$$

Where θ_U is the 100(1-a)% upper confidence limit for θ (MTBF).

Background

The engine for a light armored vehicle must have a 0.90 probability of completing 100,000 miles without an operational durability failure. In order to evaluate durability, four vehicles are tested. Each vehicle is operated until a durability failure occurs or **until** it successfully completes 100,000 miles of operation without experiencing an operational durability failure.

Determine

1. If no failures occur, what confidence do we have that the requirement has been met or exceeded?

2. If 1 failure occurs, what confidence do we have that the probability is at least 0.75?

3. If 2 failures occur, what confidence **do** we have that the probability is at least 0.50?

Solution

1. Since no failures have occurred, the number of successes is 4. We use equation 7.1 with

$$n = 4$$

$$S = 4$$

$$R_L = 0.90.$$

The confidence is:

$$\sum_{k=0}^{3} \binom{4}{k}(0.9)^k(0.1)^{4-k} = \binom{4}{0}(0.9)^0(0.1)^4 + \binom{4}{1}(0.9)^1(0.1)^3$$

$$+ \binom{4}{2}(0.9)^2(0.1)^2 + \binom{4}{3}(0.9)^3(0.1)^1$$

$$= (1)(0.0001) + (4)(0.0009) + (6)(0.0081) + (4)(0.0729)$$

$$= 0.0001 + 0.0036 + 0.0486 + 0.2916 = 0.3439.$$

We are 34% confident that the reliability meets or exceeds 0.90.

2.   The number of successes is 3.   We use equation 7.1 with

   n = 4

   S = 3

   R_L = 0.75.

The confidence is:

$$\sum_{k=0}^{2} \binom{4}{k}(0.75)^k(0.25)^{4-k} = 0.2617.$$

We are 26% confident that the reliability meets or exceeds 0.75.

3.   The number of successes is 2.   We use equation 7.1 with

   n = 4

   S = 2

   $_L$ = 0.5.

The confidence is:

$$\sum_{k=0}^{2} \binom{4}{k}(0.5)^k(0.5)^{4-k} = 0.3125.$$

We are 31% confident that the reliability meets or exceeds 0.50.

Commentary

It is interesting to note that with the small sample size, we can only reach 34% confidence that the requirement has been met or exceeded, even though we encountered zero failures. In many cases, durability requirements are impossible to demonstrate at high confidence levels because sample sizes are almost always constrained to be small.

Background

A launcher for a medium range anti-tank missile has been tested. of 100 missiles, 95 were launched successfully.

Determine

1. Point estimate of reliability.

2. Construct a 90% upper limit on the true proportion of failures using the Poisson/exponential approximation.

3. Construct an 80% confidence interval on the true reliability using the Poisson/exponential approximation.

Solution

1. Point estimate of p, the true proportion of failures is 5/100 = 0.05. Consequently, the point estimate for the reliability, R, is

$$\hat{R} = 1 - \hat{p} = 1 - 0.05 = 0.95.$$

2. We set T=n=100, r= s = 5, and α = 0.10. The approximate 90% upper limit for p, the true proportion of failures, is obtained by first determining a 90% <u>lower</u> limit for 0. The 90% lower limit for θ is

$$\theta \geq \theta_L$$

$$\geq \frac{2T}{\chi^2_{\alpha,2r}}$$

$$\geq \frac{2(100)}{15.99}$$

$$\geq 12.51.$$

Consequently, the 90% upper limit for p is

$$p \leq p_U$$

$$\leq \frac{1}{\theta_L}$$

$$\leq \frac{1}{12.51}$$

$$\leq 0.08.$$

Thus, we are 90% confident that the true proportion of failures does not exceed 0.08.

3. Weset T=n=1OO, r=s =5, **and** α = 0.20. The 80% interval for R, the launcher reliability, is obtained by first determining an 80% interval for θ. The 80% interval for θ is

$$\theta_L \leq \theta \leq \theta_U$$

$$\frac{2T}{\chi^2_{\alpha/2,2r}} \leq \theta \leq \frac{2T}{\chi^2_{1-\alpha/2,2r}}$$

$$\frac{2(100)}{18.31} \leq \theta < \frac{2(100)}{3.94}$$

$$10.92 \leq \theta \leq 50.76.$$

Consequently, an 80% interval for p, the true proportion of failures is

$$p_L \leq p \leq p_U$$

$$\frac{1}{\theta_U} \leq p \leq \frac{1}{\theta_L}$$

$$\frac{1}{50.76} \leq p \leq \frac{1}{10.92}$$

$$0.02 \leq p \leq 0.09.$$

The 80% interval for the reliability, R, is

$$R_L \leq R \leq R_U$$

$$1 - p_U \leq R \leq 1 - p_L$$

$$1 - 0.09 \leq R \leq 1 - 0.01$$

$$0.91 \leq R \leq 0.98.$$

We are 80% confident that the true launcher reliability is between 0.91 and 0.98.

Background

A new missile system has been under development and is ready for production. The contract specifies that the producer must demonstrate a proportion of successes at least equal to 0.85 (SV). The user will accept as a minimum a demonstration of at least 0.70 (MAV). An initial production test of 30 firings was conducted for the missile system, and 6 missiles fired improperly.

Determine

1.   What is our best single value estimate for the true proportion of failures?

2.   Construct exact 90%, 95%, and 99 .5% lower confidence limits for the true proportion of failures.

3.   Construct exact 90%, 95%, and 99. 5% upper confidence limits for the true proportion of failures.

4.   Construct approximate 60%, 70%, 80%, and 90% two-sided confidence limits for the true proportion of failures, using the normal approximation to the binomial.

5.   Provide an accept/reject criterion which permits the greatest number of acceptable failures which still meets a consumer' s risk of no more than 10%. What is the producer' s risk for this criterion? Is the system acceptable under this criterion?

6.   Increase the sample size to 40 and 50.   Provide an accept/reject criterion to meet a producer' s risk of 15%. What is the consumer' s risk for each criterion?

Solutions

1.   Point estimate: 6/30 = 0.20.   This corresponds to an 80% reliability.

2.   Lower confidence limits: Use Appendix B, Table 4.

     a.   **90%**   Lower limit, n = 30, s = 6.
           Lower limit = 0.109.

     b.   95%   Lower limit, n = 30, s = 6.
           Lower limit = 0.091.

     c.   99.5% Lower limit, n = 30, s = 6.
           Lower limit = 0.054.

Note that the three solutions above are lower confidence limits on the true proportion of failures, i.e. , lower limits on unreliability. If we subtract

any of the lower limits from 1, we obtain an upper limit on reliability. To convert the 90% lower limit on unreliability (0.109) to an upper limit on reliability, we subtract it from 1, i.e., 1 - 0.109 = ().891. This means that we are 90% confident that the true reliability does not exceed 0.891.

3.   Upper confidence limits: Use Appendix B, Table 4.

    a.   90%   Upper limit, n = 30, s = 6.
                Upper limit = 0.325.

    b.   95%   Upper limit, n = 30, s = 6.
                Upper limit = 0.357.

    c.   99.5% Upper limit, n = 30, s=6.
                Upper limit = 0.443.

Note that the three solutions above are upper confidence limits on the true proportion of failures, i.e., upper limits on unreliability. To obtain a lower limit on reliability, we subtract the corresponding upper limit on unreliability from 1. The 90% lower limit on reliability is thus: 1 - 0.325 = 0.675. This means that we are 90% confident that the true reliability exceeds 0.675.

4.   Approximate two-sided .limits (normal), for $\hat{p}$ = s/n = 6/30 = 0.2:

$$\text{Lower limit} = \hat{p} - z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$

$$\text{Upper limit} = \hat{p} + z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}$$

Note that the values for $z_{\alpha/2}$ can be found in Appendix B, Table 2, To use the table for two-sided limits, we convert the confidence percentage (say 60%) to a value for α(0.40), divide that value by 2($\alpha/2$ = 0.20), and locate the value for $z_{\alpha/2}$ ($z_{0.20}$ = 0.84).

    a.   60% $\alpha = 0.40$ $z_{\alpha/2} = z_{0.20} = 0.84$

                Lower limit = 0.139
                Upper limit = 0.261

    b.   70% $\alpha = 0.30$ $z_{\alpha/2} = z_{0.15} = 1.04$

                Lower limit = 0.124
                Upper limit = 0.276

    c.   80% $\alpha = 0.20$ $z_{\alpha/2} = z_{0.10} = 1.28$

                Lower limit = 0.107
                Upper limit = 0.293

d. 90% $\alpha = 0.10$ $z_{\alpha/2} = z_{0.05} = 1.645$

        Lower limit = 0.080
        Upper limit = 0.320

5. a. Use Appendix B, Table 1, n = 30. The probability of 5 or fewer failures when p is 0.3 is 0.0766. (Recall that p = 0.3 corresponds to a reliability of 0.7.) The probability of 6 or fewer failures when p is 0.3 is 0.1595. Because the consumer's risk is not to exceed 10%, we must make our decision criterion to accept with 5 or fewer failures and reject with more than 5 failures. The decision criterion to accept with 6 or fewer failures results in a consumer's risk of 15.95%, which exceeds the requirement of a 10% consumer's risk. Note that the actual consumer's risk for the criterion to accept with 5 or fewer failures is 7.66%.

    b. Use Appendix B, Table 1, n = 30. The producer's risk is the probability of rejecting the system when it has met the specification of 0.15 proportion of failures (i.e. , a reliability of 0.85) . We reject the system if 6 or more failures occur. The probability of 6 or more failures is the difference between 1 and the probability of 5 or fewer failures. The probability of 5 or fewer failures when p is 0.15 is 0.7106. Consequently, the producer's risk is 1 - 0.7106 or 0.2894 (28.94%) .

    c. The system is not acceptable because in fact more than 5 failures occurred.

6. a. Appendix B, Table 1, n = 40, p = 0.15. Producer's risk must not exceed 0. 15.

r	P(r or fewer failures)	P(r+1 or more failures)
7	0.7559	0.2441
8	0.8646	0.1354

The criterion is to reject if 9 or more failures occur; otherwise, accept.

The consumer's risk, the probability of accepting the system when, in fact, it has fallen below the MAV of 0.7, is the probability that 8 or fewer failures occur when the true proportion of failures, p, is 0.3. This value is 0.1110. Thus , there is an 11. 1% chance of accepting a bad system with this plan.

    b. Appendix B, Table 1, n = 50, p = 0.15. Producer's risk must not exceed 0. 15.

r	P(r or fewer failures)	P(r+1 or more failures)
9	0.7911	0.2089
10	0.8801	0.1199

The criterion is to reject if 11 or more failures occur; otherwise, accept.

The consumer' s risk (note above definition) is the probability that 10 or fewer failures occur when p is 0.3. This value is 0.0789. Thus , there is a 7 .89% chance of accepting a bad system with this plan.

Note that for a fixed producer's risk (approximately 13%) , the consumer's risk decreases as the sample size increases. **An** increased sample size will also result in a decreased producer's risk when the consumer's risk is held approximately constant.

Background

Two contractors are competing for a contract to produce an electronic guidance system.   Twenty-five units from Contractor I and thirty units from Contractor 2 have been tested.   The results of the test are: Contractor 1 had 2 failures, Contractor 2 had 7 failures.

Determine

1.   What is the point estimate of the difference in the true proportions of failures for the two contractors?

2.   What is the point estimate of the ratio of the true proportions of failures for the two contractors?

3.   Construct an approximate 90% lower confidence limit for the difference in the true proportions.

4.   Construct an approximate 90% lower confidence limit for the ratio of the true proportions.

5.   What confidence do we have that Contractor 1's system is at least twice as bad as Contractor 2's?   At least 50% worse than Contractor 2's?

Solutions

1. $\hat{p}_1 - \hat{p}_2 = 7/30 - 2/25 = 0.233 - 0.080 = 0.153$.

2. $\hat{p}_1 / \hat{p}_2 = 7/30 \div 2/25 = (7)(25)/2(30) = 2.91$.

3.   Lower limit $= \hat{p}_1 - \hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_1)/n_1 + \hat{p}_2(1-\hat{p}_2)/n_2}$

$$= 0.153 - z_\alpha \sqrt{(0.233)(0.767)/30 + (0.08)(0.92)/25}$$

For a 90% lower limit, $\alpha = 0.10$ and $z_\alpha = 1.28$.   (See Appendix B, Table 2.)

The lower limit for the difference in true proportions is 0.031. This means that we are 90% confident that the difference in the true proportions of failures is at least 0.031.

4.   Lower limit $= \hat{p}_1 / \hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2 \hat{p}_2^2}$

$$= 2.91 - z_\alpha \sqrt{(0.233)(0.92)/25(0.08)2} \ .$$

For a 90% lower limit, $\alpha = 0.10$ and $z_\alpha = 1.28$. The lower limit is thus 1.43. This means that we are 90% confident that Contractor 1's system is 1.43 times worse than Contractor 2's system.

5.   To find the confidence that Contractor 1's system is at least twice as bad as Contractor 2's, we must find the confidence associated with a lower limit of 2 for the ratio. Since, by definition, the lower limit is

$$\hat{p}_1/\hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

We set this expression equal to 2, and solve for z_α to obtain

$$z_\alpha = (\hat{p}_1/\hat{p}_2 - 2)/\sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

Substituting 0.233 for $\hat{p}_1$, 0.08 for $\hat{p}_2$, and 25 for n_2, we have

$$z_\alpha = 0.788 \ .$$

We look in Appendix B, Table 2 to find the value of α which corresponds to z_α = 0.788. Since, by definition, $P(Z \geq z_\alpha) = \alpha$, the desired value of α is located under the column labeled $P(Z \geq z_\alpha)$. Thus the value of α is 0.215. This represents a $100(1-(Y)\% = 78.5\%$ lower confidence limit, so we are 78.5% confident that Contractor 1's system is at least twice as bad as Contractor 2's.

To find the confidence that Contractor 1's system is at least 50% worse than Contractor 2's, we solve the following equation for z_α,

$$\hat{p}_1/\hat{p}_2 - z_\alpha \sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ = \ 1.5 \ .$$

The solution is:

$$z_\alpha = (\hat{p}_1/\hat{p}_2 - 1.5)\sqrt{\hat{p}_1(1-\hat{p}_2)/n_2\hat{p}_2^2} \ .$$

Substituting 0.233 for $\hat{p}_1$, 0.08 for $\hat{p}_2$, and 25 for n_2, we have

$$z_\alpha = 1.22 \ .$$

This corresponds to an α of 0.1112, so we are 88.88% confident that Contractor 1's system is at least 50% worse than Contractor 2's.

Background

Six electronic systems were tested. All systems had the same configuration, and no design changes were introduced during the test. The test experience is tabulated below.

System	System Age at Start of Test, Hrs.	System Age at Failure(s) Hrs.	System Age at End of Test, Hrs.
1	0	13, 37, 60	275
2	75	154	290
3	0	73	290
4	150	190, 218	270
5	0	3, 52, 227	260
6	0	39, 166, 167, 209	260

Determine

Is exponential data analysis appropriate for this data set?

Solution

An average failure rate plot will be used to determine if there is a trend to the data. The following graph, although not a necessary part of the analysis, is included to aid visualization of the data.
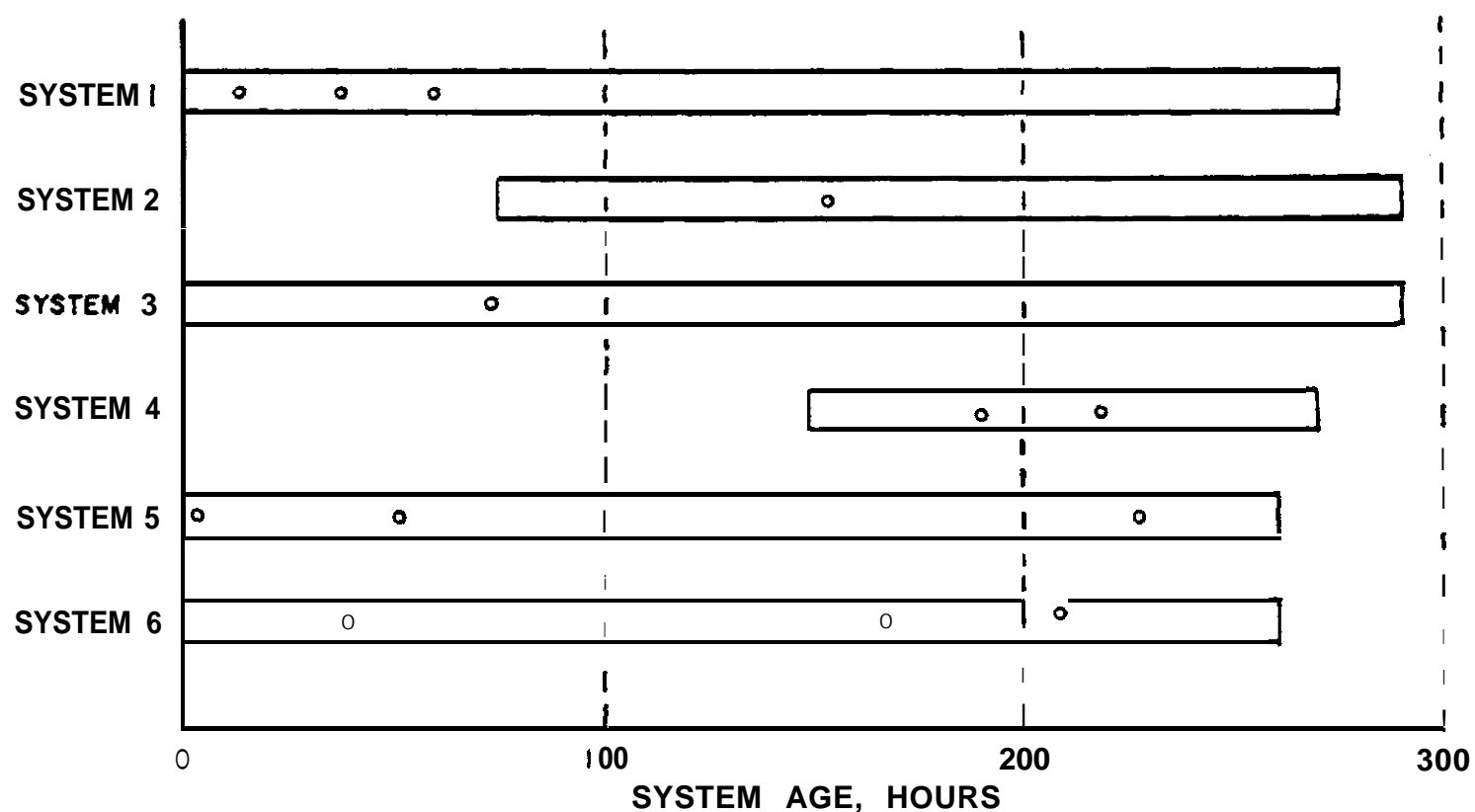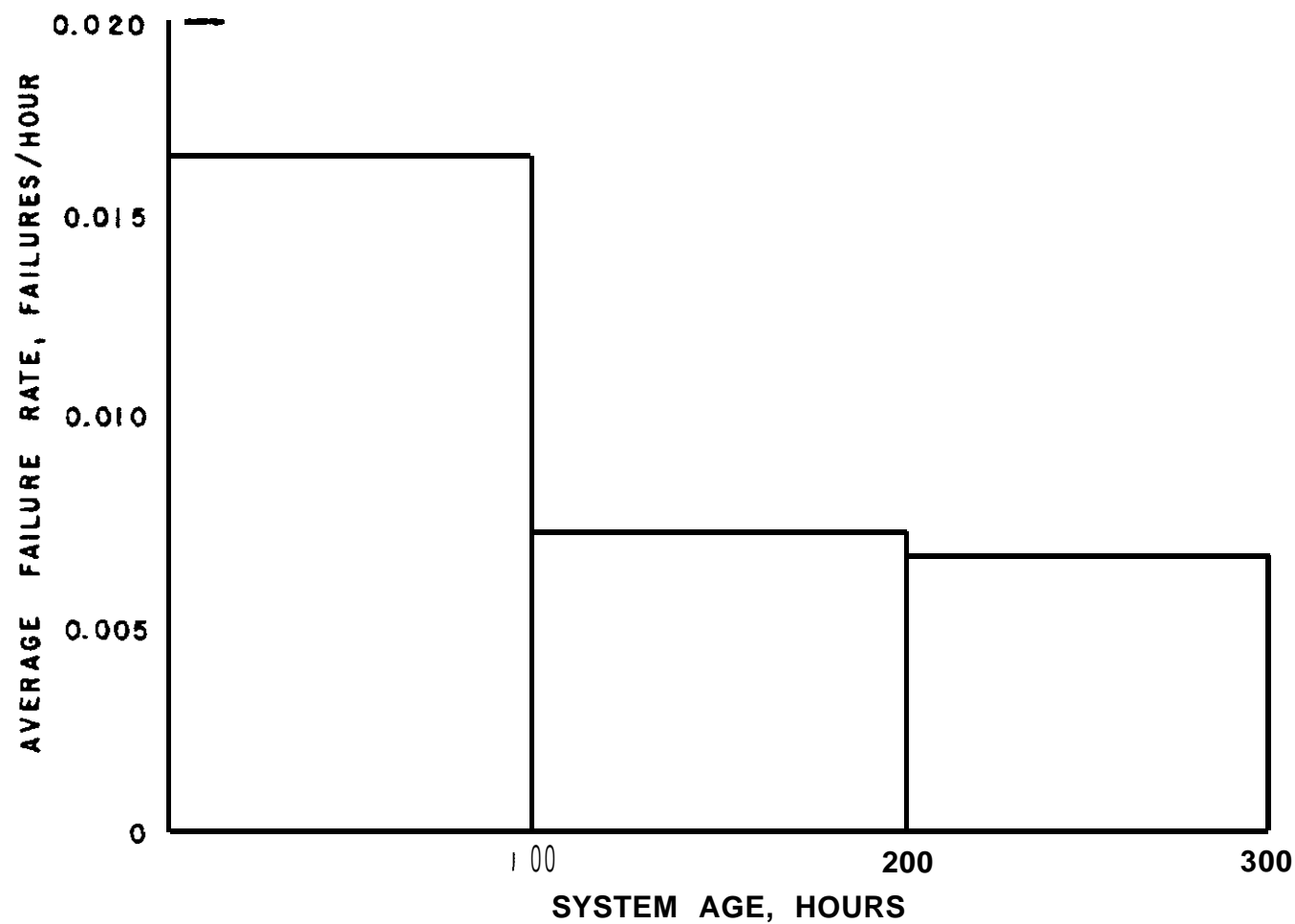

The data will be broken down into three equal intervals. The steps involved

in arriving at the average failure rate for each **interval** are contained in the
following table.

Interval	Failures	Operating Hours	Average Failure Rate
o-1oo	7	425	7/425 = 0.0165
100-200	4	550	4/550 = 0.0073
200-300	3	445	3/445 = 0.0067

These average failure rates are plotted on the following graph.

The average failure rate plot suggests very strongly that there is a decreas-
ing failure rate as the system ages, and exponential data analysis should not
be used unless, at a minimum, a caveat about the decreasing failure rate is
included.

Commentary

1.   Although this is a fictional data set, the pattern to the data is fre-
quently observed in real data sets.

2.   For a data set of this type, it is generally useful to consider the
actual failure types and corrective actions encountered.   This tends to
clarify how permanent the high initial failure rate might be.

Background

Three developmental vehicles were operated under test conditions that closely matched the operational mode summary and mission profile for the system. All vehicles had the same physical configuration. Only one relatively minor design change was introduced to the test vehicles during the test. Scoring of **test incidents** determined that there were 7 operational-mission failures. 'The following table displays the operational mission failure data.

| Vehicle Number	Odometer at Start(km)	Odometer at Failure	Odometer at End (km)
1	0	None	6,147
2	0	3,721; 6,121; 6,175 9,002	11,000
3	0	216; 561; 2,804	5,012

Determine

Is exponential data analysis appropriate for this data set?

Solution

An average failure rate plot will be used to determine if there is a trend to the data. Three equal intervals will (arbitrarily) be used.

Interval	Failures	Kilometers	Average Failure Rate
0-4,000	4	12,000	4/12,000 = 0.00033
4,000-8,000	2	7,159	2/7,159 = 0.00028
8,000-12,000	1	3,000	1/3,000 = 0.00033

These average failure rates are plotted on the following graph.
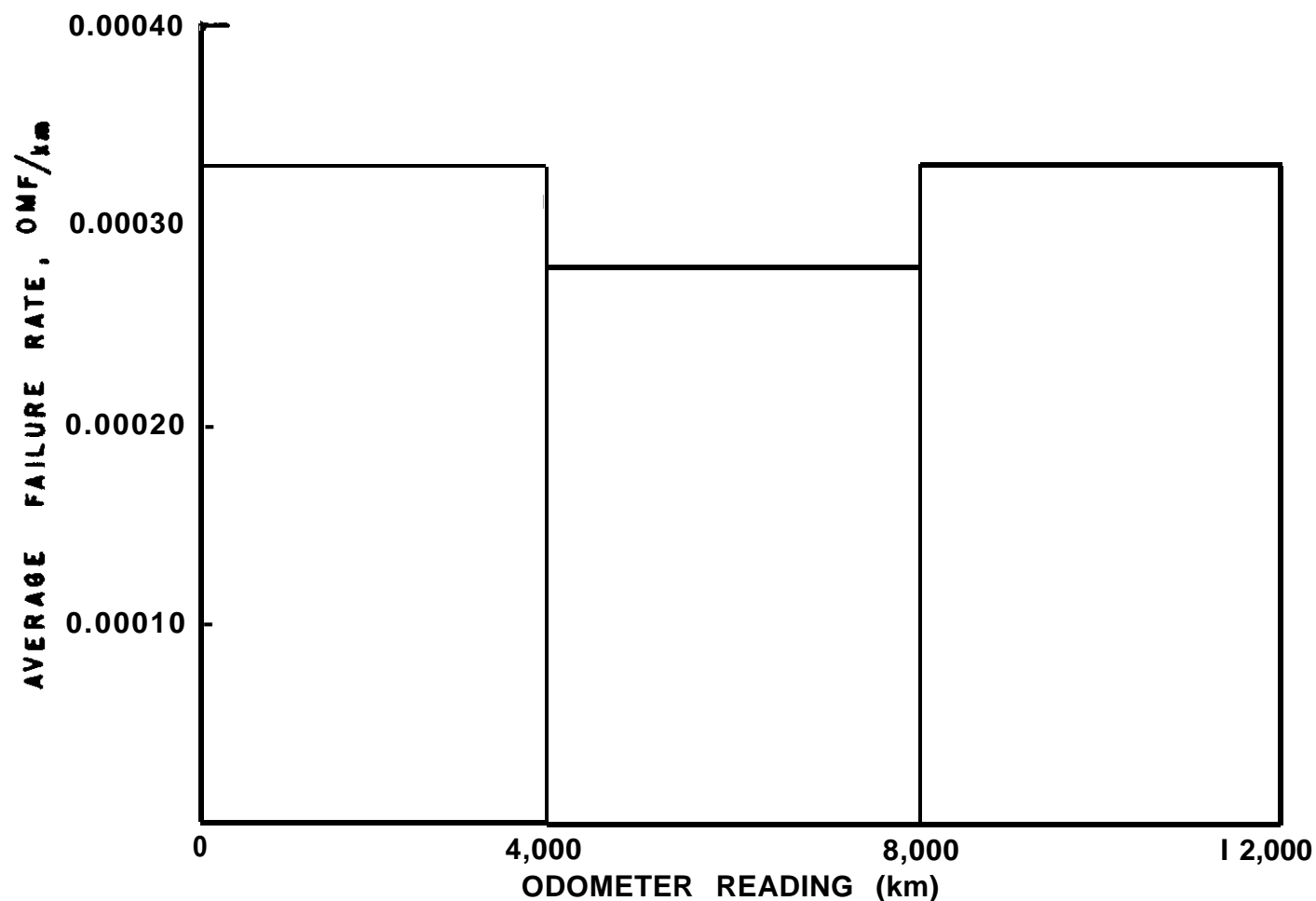
Since the average failure rate plot is essentially horizontal, there is virtually no evidence of trend in the data, and exponential data analysis procedures may be used.

Commentary

For large data sets , the average failure rate plot gives a very precise picture of the actual failure rate pattern. For small data sets, such as this one, chance plays a very heavy role. For example, in this case we observed one failure in the last interval. Just one more, or one less failure in this interval would make a drastic difference in the observed average failure rate. More formal trend tests address whether such variations could reasonably be due to chance alone.

CASE STUDY NO. 7-7

Background

The vehicle system discussed in Case Study 7-6 has a mission duration of 100 kilometers. The user has stated a minimum acceptable value (MAV) of 2,000 mean kilometers between operational mission failure (MKBOMF) and the contractual reliability requirement is equivalent to a 4,000 MKBOMF specified value (SV). The analysis in Case Study 7-6 showed no trend to the data. The test gave 22,159 km of exposure, and 7 operational mission failures were observed.

Determine

1. Point estimate of MKBOMF, mission reliability and failure rate.
2. 80% confidence interval for MKBOMF and mission reliability.
3. 80% lower confidence limit for MKBOMF.
4. 80% upper confidence limit for MKBOMF.
5. What confidence do we have that the MAV has been met or exceeded?
6. What confidence do we have that the SV has not been obtained?
7. Does the statistical evidence suggest that the reliability is satisfactory, or not?

Solutions

Because Case Study 7-6 gave no evidence of trend, exponential data analysis procedures will be used. Note that they are all based on test exposure, T = 22,159 kilometers, and number of failures, r = 7. Actual odometer readings at failure need not be considered, except to note that the test exposure is "time" terminated.

1. Point estimates of 0, R(100), and λ.

    a. Apply equation 7.6

$$\hat{\theta} = \frac{T}{r} = \frac{22,159}{7} = 3165.6 \text{ MKBOMF}$$

    Convert to mission reliability using equation 7.7:
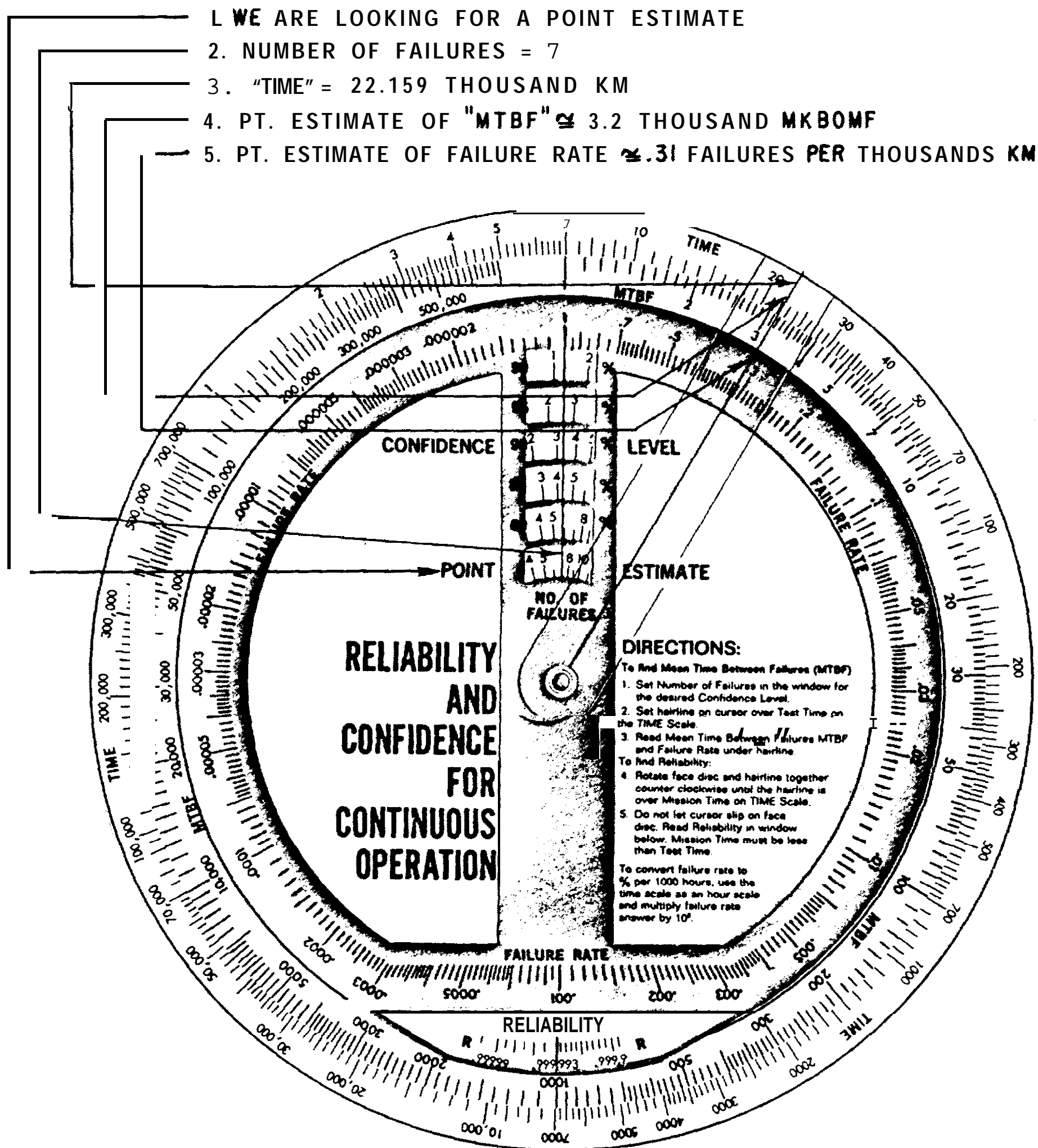
$$\hat{R}(x) = e^{-x/6}$$

$$\hat{R}(100) = e^{-100/3165.6} = e^{-0.0316} = 0.969.$$

    Convert to failure rate using equation 7.8:

$$\hat{\lambda} = \frac{1}{\hat{\theta}} = \frac{1}{3165.6} = 0.000316 \text{ failures per km.}$$
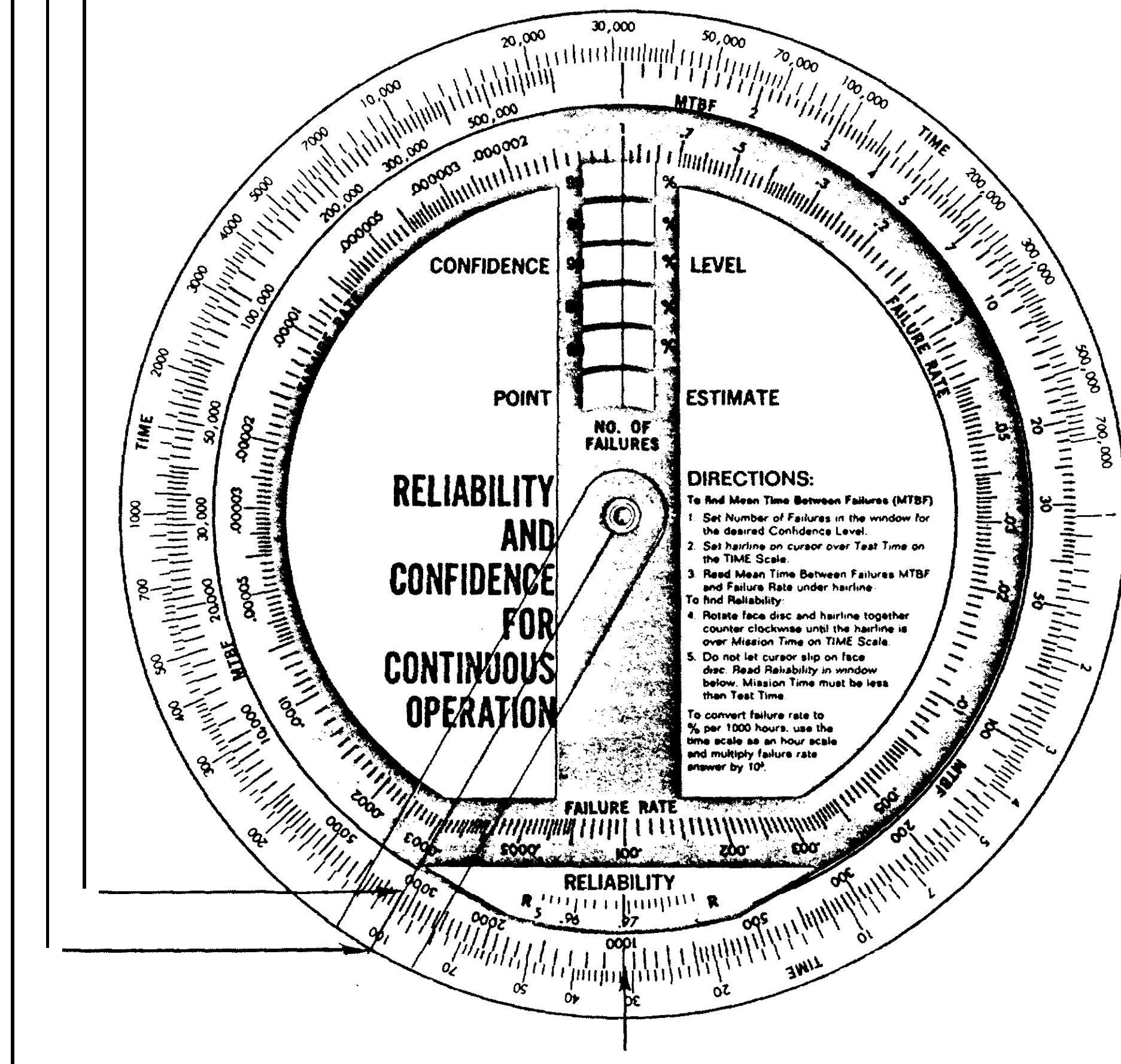
    b. Use a reliability computer.

The next two figures illustrate the two-step solution procedure for part 1 using the reliability computer.

L WE ARE LOOKING FOR A POINT ESTIMATE

2. NUMBER OF FAILURES = 7

3. "TIME" = 22.159 THOUSAND KM

4. PT. ESTIMATE OF "MTBF" ≅ 3.2 THOUSAND MKBOMF

5. PT. ESTIMATE OF FAILURE RATE ≅ .31 FAILURES PER THOUSANDS KM

6. PT. ESTIMATE OF "MTBF" ≅ 3,200 MKBOMF
7. "TIME" = MISSION DURATION = 100 KM
8. PT. ESTIMATE OF RELIABILITY ≅ .969

RELIABILITY AND CONFIDENCE FOR CONTINUOUS OPERATION

CONFIDENCE LEVEL

POINT ESTIMATE

NO. OF FAILURES

DIRECTIONS:

To Find Mean Time Between Failures (MTBF)

1. Set Number of Failures in the window for the desired Confidence Level.
2. Set hairline on cursor over Test Time on the TIME Scale.
3. Read Mean Time Between Failures MTBF and Failure Rate under hairline.

To find Reliability:

4. Rotate face disc and hairline together counter clockwise until the hairline is over Mission Time on TIME Scale.
5. Do not let cursor slip on face disc. Read Reliability in window below. Mission Time must be less than Test Time.

To convert failure rate to % per 1000 hours, use the time scale as an hour scale and multiply failure rate answer by 10⁵.

2. An 80% confidence interval for θ and reliability for a 100-kilometer mission R(100).

a. Using Table 8, Appendix B we obtain the confidence limit multipliers for the case of 7 failures and 80% confidence interval, i.e., 90% upper and 90% lower confidence limits. These multipliers are 0.665 and 2.797 for the 90% lower and upper confidence limits, respectively. Note we use Table 8a because this is a kilometers (i.e., time) terminated test.

$$\theta_L = \text{multiplier} (\hat{\theta}) = (0.595)(3165.6) = 1883.5 \text{ MKBOMF}$$

$$\theta_u = \text{multiplier} (\hat{\theta}) = (1.797)(3165.6) = 5688.6 \text{ MKBOMF}$$

We are therefore 80% confident that

$$1883.5 \leq \theta \leq 5688.6 \text{ MKBOMF}$$

b. Using inequality 7.9a, we find, for α = 0.20,

$$\frac{2T}{\chi^2_{\alpha/2,2r+2}} \leq \theta \leq \frac{2T}{\chi^2_{1-\alpha/2,2r}}$$

$$\frac{2(22,159)}{\chi^2_{0.10,16}} \leq \theta \leq \frac{2(22,159)}{\chi^2_{*0.90,14}}$$

Using Appendix B, Table 5 for the appropriate χ^2 values, we have

$$\frac{44,318}{23.55} \leq \theta \leq \frac{44,318}{7.79}$$

We are 80% confident that

$$1881.9 \leq \theta \leq 5689.0 \text{ MKBOMF}$$

In other words, we are reasonably sure that the MKBOMF is not less than 1881.9, nor greater than 5689.0.

c. Converting to mission reliability using inequality 7.16, we find

$$e^{-x/\theta_L} \leq R(x) \leq e^{-x/\theta_U}$$

$$e^{-100/1881.9} \leq R(100) \leq e^{-100/5689.0}$$

We are 80% confident that

$$0.948 \leq R(100) \leq 0.983$$

<u>Commentary</u>

The reliability computer cannot be used for confidence intervals since it does not have a capability for upper limits.

3.   An 80% lower confidence limit for θ.

   a.   Use Table 8a, Appendix B to find the multiplier for *an 80% lower* confidence limit with 7 failures.

$$\theta_L = \text{multiplier } (\hat{\theta}) = (0.684)(3165.6) = 2165.3 \text{ MKBOMF}$$

   Therefore, we are 80% confident that

$$\theta \geq 2165.3 \text{ MKBOMF}$$

   b.   Using inequality 7.10a, we find

$$\theta \geq \frac{2T}{\chi^2_{\alpha,\, 2r+2}}$$

$$\theta \geq \frac{2(22,159)}{\chi^2_{0.20,16}}$$

   Using Appendix B, Table 5 for the χ^2 value, we have

$$\theta \geq \frac{44,318}{20.47}$$

   We are 80% confident that
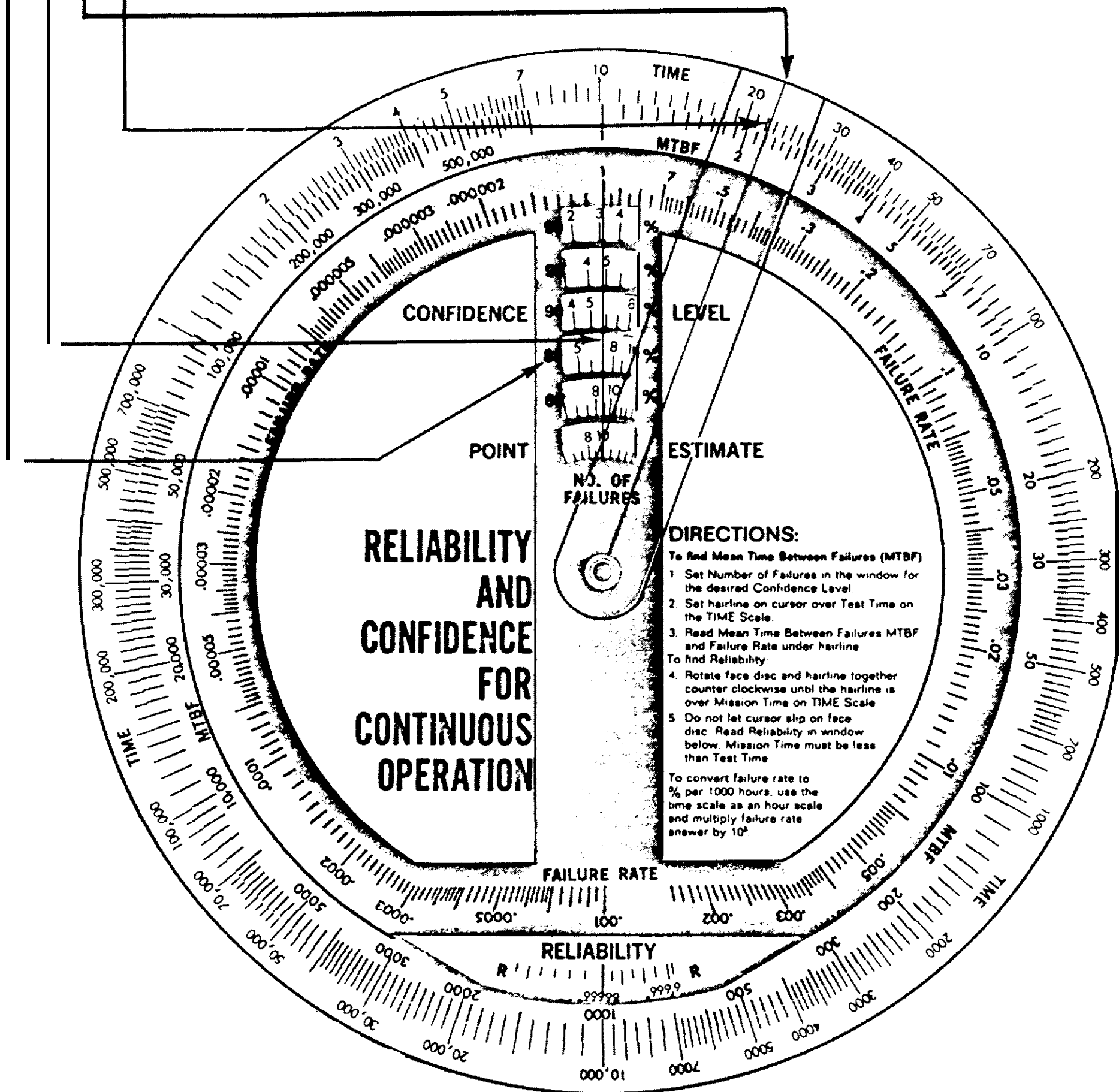
$$\theta \geq 2165.0 \text{ MKBOMF}$$

   c.   Using a reliability computer, we find

1. CONFIDENCE LEVEL = 80 %
2. NUMBER OF FAILURES = 7
3. "TIME" = 22.159 THOUSAND KM
4. LOWER LIMIT ≅ 21.7 THOUSAND KM

4. An 80% upper confidence limit for 0.

    a. Use Table 8a, Appendix B to find the multiplier for an 80% upper confidence limit with 7 failures.

$$\theta_U = \text{multiplier } (\hat{\theta}) = (1.479)(3165.6) = 4681.9 \text{ MKBOMF}$$

Therefore, we are 80% confident that

$$\theta \leq 4681.9 \text{ MKBOMF}$$

    b. Using inequality 7.11, we find

$$\theta \leq \frac{2T}{\chi^2_{1-\alpha,2r}}$$

$$\theta \leq \frac{2(22,159)}{\chi^2_{0.80,14}}$$

Using Appendix B, Table 5 for the χ^2 value, we find

$$\theta \leq \frac{44,318}{9.47}$$

We are 80% confident that

$$\theta \leq 4679.8 \text{ MKBOMF}$$

Commentary

The reliability computer does not have a capability for upper limits.

5. What confidence do we have that $\theta \geq 2,000$?

    a. Using equation 7.12a, we find

$$\chi^2_{\alpha,2r+2} = \frac{2T}{\theta_L}$$

$$\chi^2_{(x,16} = \frac{2(22,159)}{2000} = 22\ 159$$

Searching Appendix B, Table 5 in the row labeled 16 for a value of 22.159, we find values of 20.47 and 23.55. Interpolating, we obtain $\alpha \cong 0.14$. Confidence is $100(1-cY)\% \cong 100(1-0.14)\%$. We are approximately 86% confident that

$$\theta \geq 2000 \text{ MKBOMF}$$

b. Using equation 7.13a, we find

$$\alpha = \sum_{k=0}^{r} \frac{(T/\theta_L)^k \, e^{-(T/\theta_L)}}{k!}$$

$$T/\theta_L = \frac{22,159}{2000} = 11.0795$$

$$r = 7$$

$$\alpha = \frac{(11.0795)^0 e^{-11.0795}}{1} + \frac{(11.0795)^1 e^{-11.0795}}{1}$$

$$+ \frac{(11.0795)^2 e^{-11.0795}}{2} + \frac{(11.0795)^3 e^{-11.0795}}{6}$$

$$+ \frac{(11.0795)^4 e^{-11.0795}}{2 \quad 4} + \frac{(11.0795)^5 e^{-11.0795}}{120}$$

$$+ \frac{(11.0795)^6 e^{-11.0795}}{720} + \frac{(11.0795)^7 e^{-11.0795}}{5040}$$

$$= 0.0000 + 0.0002 + 0.0009 + 0.0035 + 0.0097$$

$$+ 0.0215 + 0.0396 + 0.0627$$

$$= 0.1381$$

We are **86.2%** confident that

$$\theta \geq 2000 \text{ MKBOMF}$$

6. What confidence do we have that 6 < 4,000?

The confidence that $\theta < 4,000$ is the same as the confidence that $\theta < 4,000$. The former statement is easier to interpret, although the latter is the usual expression.

a. Using equation 7.14a, we find

$$\chi^2_{1-\alpha, 2r} = \frac{2T}{\theta_U}$$

$$\chi^2_{1-\alpha, 14} = \frac{2(22,159)}{4000} = 11.08$$

Searching Appendix B, Table 5 in the row labeled 14 for a value of 11.08, we find values of **10.16** and 13.34. Interpolating, we obtain l-a ≅ 0.68. Confidence is 100(1-(Y)% ≅ 100(0.68)%. We are approximately 68% confident that

$$\theta < \textbf{4000 MKBOMF}$$

b.   Using equation 7.15, we obtain

$$1-\alpha = \sum_{k=0}^{r-1} \frac{(T/\theta_U)^k \, e^{-(T/\theta_U)}}{k!}$$

$$T/\theta_U \cong \frac{22,159}{4000} = 5.53975$$

$$r-1 = 6$$

$$1-\alpha = \frac{(5.53975)0e^{-5''53975}}{1} + \frac{(5.53975)^1 e^{-5.53975}}{1}$$

$$+ \frac{(5.53975)^2 e^{-5.53975}}{2} + \frac{(5.53975)3e^{-5-53975}}{6}$$

$$+ \frac{(5.53975)^4 e^{-5.53975}}{24} + \frac{(5.53975)5e^{-5-53975}}{\textbf{120}}$$

$$+ \frac{(5.53975)^6 e^{-5.53975}}{720}$$

$$= 0.0039 + 0.0218 + 0.0603 + 0.1113 + 0.1541 + 0.1708$$

$$+ 0.1577$$

$$= 0.6798$$

We are **68%** confident that

$$\theta < 4000 \text{ \textbf{MKBOMF}}$$

7.   Does the reliability appear satisfactory?

We are **86%** confident that the user's needs have been met, but only 68% confident that contractual obligations were not met. There is stronger evidence that the reliability is satisfactory than not. If many more failures were experienced, we would have low confidence that the user's needs *were* met, and we would also have higher confidence that the contractual obligations were not met, suggesting that the reliability is not satisfactory from both standpoints.

Background

An avionics system has the following reliability requirements: the minimum acceptable value (MAV) = 150 hrs. MTBF, and the specified value (SV) = 450 hrs. MTBF. Three of these systems were tested for 100 hours (each) under test conditions that closely duplicated the expected operational environment. No failures were observed during this test.

Determine

An 80% lower confidence limit for MTBF, and the confidence that the MAV has been attained.

Commentary

The case of a test with zero failures has some interesting features. With no failures, there is no way to determine the type of failure pattern. If we have some assurance that the equipment will not degrade as it ages, we can make a constant failure rate assumption, which, in turn, permits an exponential data analysis.

If we attempt to obtain a point estimate of 9, we get:

$$\hat{\theta} = \frac{T}{r} = \frac{T}{0} = \text{indeterminate}$$

Similarly, the upper limit is indeterminate. We can, however, obtain lower confidence limits.

Solutions

1.   80% lower confidence limit for 9.

   a.   Note that the technique of using the multipliers from Table 8, Appendix B, cannot be used for the case of zero failures.

   b.   Using inequality **7.10a, we** find

$$\theta \geq \frac{2T}{\chi^2_{a,\ 2r+2}}$$

   We have in this case, T = 300, α = 0.2 and r = 0, so
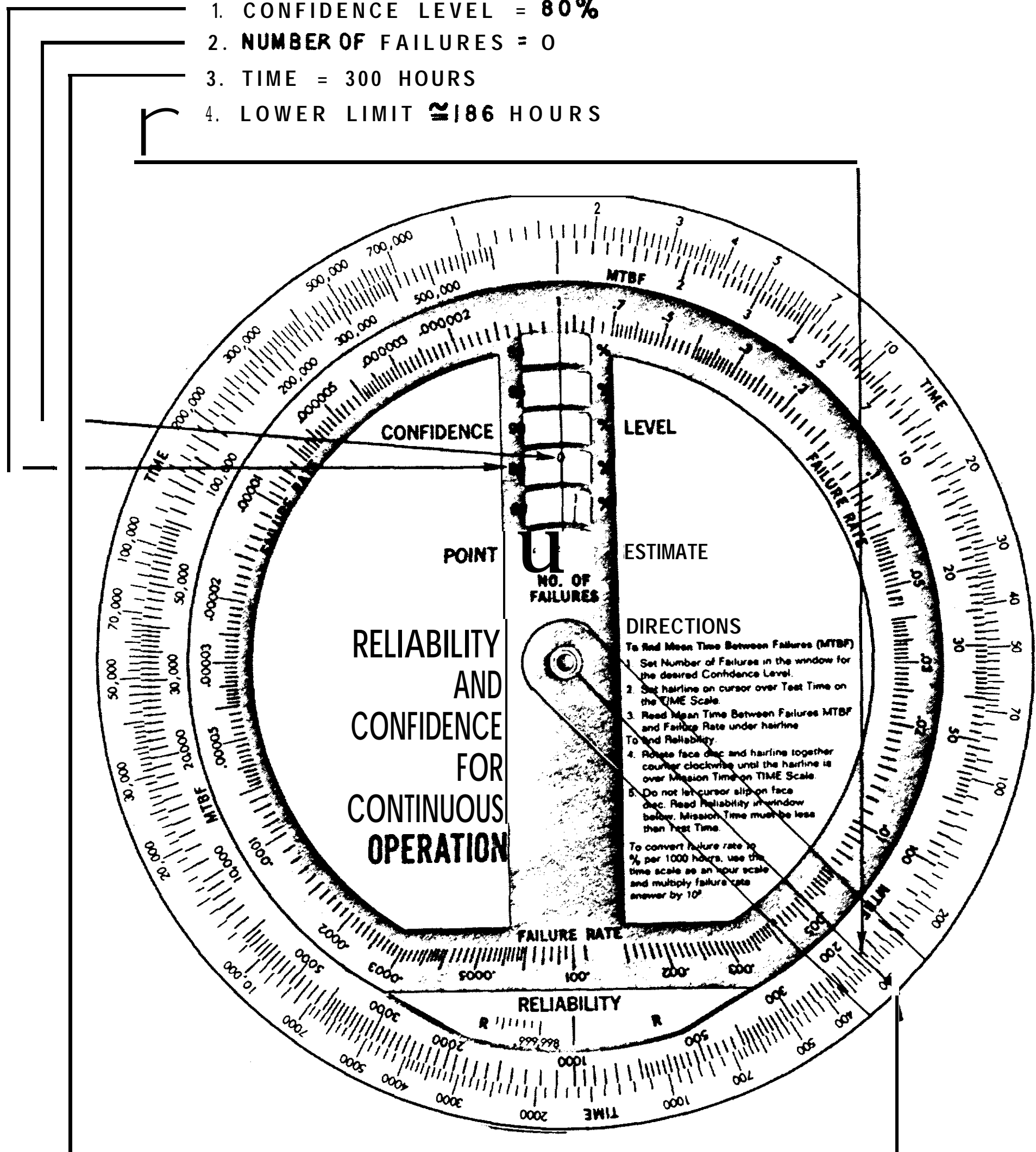
$$\theta \geq \frac{2(300)}{\chi^2_{0.2,2}}$$

Using Appendix B, Table 5 for the χ^2 value, we find

$$\theta \geq \frac{600}{3.22}$$

We are 80% confident that $\theta \geq 186.3$ hrs MTBF

b.   Using a reliability computer, we find

1. CONFIDENCE LEVEL = **80%**

2. **NUMBER OF** FAILURES = O

3. TIME = 300 HOURS

4. LOWER LIMIT $\cong$ 186 HOURS

2. Confidence that $\theta \geq 150$

    a. As noted in part 1 of this problem, Table 8 cannot be used for the case of zero failures.

    b. Using equation 7.12a, we find

    $$\chi^2_{\alpha,2r+2} \cdot \frac{2T}{\theta_L}$$

    $$\chi^2_{\alpha,2} = \frac{2(300)}{150} = 4.0$$

    Searching Appendix B, Table 5 in the row labeled 2 for a value of 4.0, we find values of 3.22 and 4.6o. Interpolating, we obtain $\alpha \cong$ .13. Confidence is $100(1-\alpha)\% = 100(1-0.13)\%$. We are approximately 87% confident that

    $$\theta \geq 150 \text{ hrs MTBF}$$

    c. Using equation 7.13a, we find

    $$\alpha = \sum_{k=0}^{r} \frac{(T/\theta_L)^k \; e^{-(T/\theta_L)}}{k!}$$

    For r = 0, this simplifies to

    $$\alpha = e^{-(T/\theta_L)}$$

    In this case,

    $$\alpha = e^{-(300/150)} = e^{-2} = 0.135$$

    We are 86.5% confident that

    $$\theta \geq 150 \text{ hrs MTBF}$$

.

Background

A system is being tested using test plan XIIC from Appendix B, Table 6. The upper test value (SV) is 100 hours MTBF, and the lower test value (MAV) is 50 hours MTBF. The required test duration is 940 hours, and 14 failures are rejectable. The source of data for this test plan is not relevant for this case study, but is presented here for future reference. Chapter 8 contains detailed discussions on the formation and use of this and other test plans.

The seventh failure has just occurred after only 57 hours of test exposure. Because of the excessive number of failures , an evaluation is to be done at this point in the test. Preliminary analysis of the data showed no evidence of trend, i.e., failure rate appeared constant,

Determine

1.   Point estimate of MTBF.

2.   80% confidence interval for MTBF.

3.   80% lower confidence limit for MTBF.

4.   80% upper confidence limit for MTBF.

5.'   What confidence do we have that the lower test value has been met or exceeded?

6.   What confidence do we have that the upper test value has not been attained?

7.   Does the statistical evidence suggest that the reliability is satisfactory or not?

Commentary

Because an evaluation is being made at this point based on what was observed, we do not have a legitimate random sample. The true risks in making decisions based on such an analysis are difficult to determine. They are, in fact, substantially higher than the ones associated with the original plan. Consequently, the following analyses are all somewhat pessimistically biased.

Solutions

Since the seventh failure has just occurred, this is failure terminated data.

1.   Point estimate of θ.

Applying equation 7.6, we obtain

$$\theta = \frac{T}{r} = \frac{57}{7} = 8.14 \text{ hrs MTBF}$$

2.   **An** 80% confidence interval for θ.

   a.   Use Table 8b, Appendix B to obtain the confidence limit multiplier
        for the case of 7 failures and 80% confidence interval, i.e., 90%
        upper and lower confidence limits. Note we are using Table **8b**
        because the test is <u>failure</u> terminated.

$$\theta_U = \text{multiplier } (\overset{..}{\theta}) = (2.797)(8.14) = 14.63 \text{ hrs. MTBF}$$

$$\theta_L = \text{multiplier } (\hat{\theta}) = (0.665)(8.14) = 5.41 \text{ hrs. MTBF}$$

   We are therefore 80% confident that

$$5.41 \leq \theta < 14.63 \text{ hrs. MTBF}$$

   b.   Using inequality **7.9b,** we find, for $\alpha = 0.20$,

$$\frac{2T}{\chi^2_{\alpha/2,2r}} \leq \theta \leq \frac{2T}{\chi^2_{1-\alpha/2,2r}}$$

   Using Appendix B, Table 5 for X^2 values:

$$\frac{114}{21.07} \leq \theta \leq \frac{114}{7.79}$$

   We are 80% confident that

$$5.41 \leq \theta \leq 14.63 \text{ hrs MTBF}$$

3.   An 80% *lower* confidence limit for θ.

   a.   Use Table 8b, Appendix B to find the multiplier for an 80% lower
        confidence limit with 7 failures.

$$\theta_L = \text{multiplier } (\hat{\theta}) = (0.771)(8.14) = 6.28 \text{ hrs. MTBF}$$

   Therefore, we are 80% confident that

$$\theta \geq 6.28 \text{ hrs. MTBF}$$

   b.   Using inequality **7.10b,** we find

$$\theta \geq \frac{2T}{\chi^2_{\alpha,2r}}$$

Using Appendix B, Table 5 for the X^2 value, we find

$$\theta \geq \frac{114}{18.15}$$

We are 80% confident that

$$\theta > 6.28 \text{ hrs MTBF}$$

4. An 80% upper confidence limit for θ.

   a. Use Table 8b, Appendix B to find the multiplier for an 80% upper confidence limit with 7 failures.

   $$\theta_U = \text{multiplier } (\hat{\theta}) = (1.479)(8.14) = 12.04 \text{ hrs. MTBF}$$

   Therefore, we are 80% confident that

   $$\theta \leq 12.04 \text{ hrs. MTBF}$$

   b. Using inequality 7.11, we find

   $$\theta \leq \frac{2T}{X^2_{1-\alpha,2r}}$$

   Using Appendix B, Table 5 for the X^2 value, we obtain

   $$\theta \leq \frac{114}{9.47}$$

   We are 80% confident that

   $$\theta \leq 12.04 \text{ hrs MTBF}$$

5. What confidence do we have that $\theta \geq 50$?

   a. Using equation 7.12b, we find

   $$X^2_{\alpha,2r} = 2T/\theta_L$$

   $$X^2_{\alpha,14} = 2(57)/50 = 2.28$$

   Searching Appendix B, Table 5 in the row labeled 14 for a value of 2.28, we find that we are beyond the end of the table, and $\alpha > 0.995$. The confidence is $100(1-\alpha)\%$, $100(1-o.995)\%$, or less than 0.5%.

   We are less than 0.5% confident that

   $$\theta \geq 50 \text{ hrs MTBF}$$

b.   Using equation 7.13b, we find

$$\alpha = \sum_{k=0}^{r-1} \frac{(T/\theta_L)^k \, e^{-(T/\theta_L)}}{k!}$$

where

$$T/\theta_L = 57/50 = 1.14 \text{ and } r - 1 = 7 - 1 = 6.$$

Solving equation 7.13b, we find $\alpha = 0.9998$.

We are 0.02% confident that $\theta \geq 50$ hours MTBF.

6.   **What** confidence do we have that $\theta \leq 100$?

a.   **Using** equation 7.14, we find

$$\chi^2_{1-\alpha,14} = \frac{2(57)}{100} \approx 1.14$$

Searching Appendix B, Table 5 in the row labeled 14 for a value of 1.14, we find that we are well beyond the end of the table, and $1-\alpha \cong 1.0$. The confidence is 100(1-o)%, 100(1.0)%, or essentially 100%.

We are essentially 100% confident that

$$\theta \leq 100 \text{ hrs MTBF}$$

b.   Using equation 7.15, we find

$$1-\alpha = \sum_{k=0}^{r-1} \frac{(T/\theta_U)^k \, e^{-(T/\theta_U)}}{k!}$$

where

$$T/\theta_U = 0.57 \text{ and } r - 1 = 7 - 1 = 6$$

Solving equation 7.15, we find $1-\alpha = 0.99999$.

We are essentially 100% confident that

$$\theta \leq 100 \text{ hrs MTBF}$$

7.   **Does** the reliability appear satisfactory?

Since we have essentially 0% confidence that the lower test value is met or exceeded, and since we have essentially 100% confidence that the upper test value is not met, there is overwhelming evidence that the reliability is not satisfactory , even taking into consideration the fact that the analysis may be somewhat pessimistically biased.

In this case, the evidence is so strong that we can even state that we are 99.98% confident that $\theta < 50$ hrs MTBF, though, ordinarily, upper limit statements are associated with the upper test value, and lower limit statements are associated with the lower test value.

Commentary

Test plan **XIIC** from Appendix B, Table 6 required a test duration of 940 hours to achieve true producer's and consumer's risks of 0.096 and 0.106, respectively. Since the system appears to be "failing miserably," the user has chosen to stop testing after 57 hours. No doubt this is a wise decision from an economic standpoint. However, the user should be fully cognizant that the risks associated with his abnormally terminated test are not **0.096** and o.1o6, nor are they the ones advertised in the preceding computations. The calculation of the true risks is well beyond the scope of this work.